APPLICATIONS

POPULATION

POPULATION GROWS AT RATE PROPORTIONAL TO EXISTING POPULATION

$$\frac{dP}{dt} = kP, \quad P(0)=P_0 \longrightarrow P = P_0 e^{kt}$$

$k>0$

AS $t \to \infty$, $P \to \infty$

BAD ASSU

EXPECT WHEN P IS LARGE k IS SMALLER $\longrightarrow$ k IS NOT A CONSTANT

k IS A DECREASING FUNCTION OF P

$$\frac{dP}{dt} = k(P)\,P$$

$$\int \frac{1}{k(P)P}\,dP = \int dt$$

SIMPLEST

$k(P) = -mP + b$

$(m>0, b>0)$

LOGISTIC POPULATION MODEL

$$\frac{dP}{dt} = (-mP+b)P = -m\left(P-\frac{b}{m}\right)P \qquad C=\frac{b}{m}>0$$

$$m,b>0 \qquad = -m(P-C)P$$

$$\frac{dP}{dt} = -m(P-C)P,\ P(0)=P_0$$

$$\int \frac{1}{(P-C)P}\,dP = \int -m\,dt$$

$$\frac{1}{(P-C)P} = \frac{A}{P-C} + \frac{B}{P}$$

$$1 = AP + B(P-C)$$

$$P=0:\ 1 = B(-C) \rightarrow B = -\frac{1}{C}$$

$$P=C:\ 1 = AC \rightarrow A = \frac{1}{C}$$

$$\int\left(\frac{\frac{1}{C}}{P-C} - \frac{\frac{1}{C}}{P}\right)dP$$

$$\frac{1}{C}\ln|P-C| - \frac{1}{C}\ln|P| = -mt+k$$

$$\ln|P-C| - \ln|P| = -mCt+k$$

$$e^{\ln\left|\frac{P-C}{P}\right|} = e^{-mCt+k}$$

$$\frac{P-C}{P} = Ke^{-mCt}$$

$$1-\frac{C}{P} = Ke^{-mCt}$$

$$1+Ke^{-mCt} = \frac{C}{P}$$

$$P = \frac{C}{1+Ke^{-mCt}}$$

$$P_0 = \frac{C}{1+K}$$

$$1+K = \frac{C}{P_0} \rightarrow K = \frac{C}{P_0}-1$$

$$P = \frac{C}{1+\left(\frac{C}{P_0}-1\right)e^{-mCt}}$$

$$P = \frac{CP_0}{P_0+(C-P_0)e^{-mCt}} \qquad m, C > 0$$

AS $t \to \infty$, $P \to \dfrac{CP_0}{P_0+(C-P_0)0} = C =$ ENVIRONMENTAL CARRYING CAPACITY A/K/A P_c

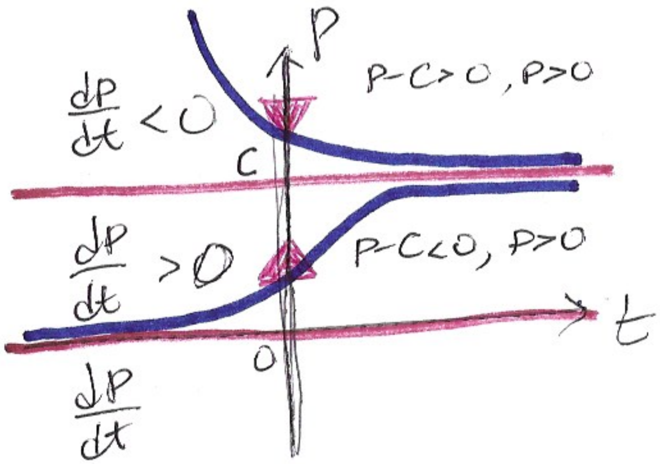
IE. $P = C$ IS A HORIZONTAL ASYMPTOTE

$$\frac{dP}{dt} = -m(P-C)P = -m(P^2-CP)$$

IS ALSO AUTONOMOUS

EQ SOL'N: $\frac{dP}{dt} = 0 \to -m(P-C)P = 0$

$P = C, 0$



WHERE IS THE INFLECTION POINT?

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\frac{dP}{dt} = \left(\frac{d}{dP}\frac{dP}{dt}\right)\left(\frac{dP}{dt}\right)$$

$$= -m(2P-C)\frac{dP}{dt} = 0$$

$2P - C = 0$ or $\frac{dP}{dt} = 0$

$\boxed{P = \frac{1}{2}C}$ $\qquad P = 0, C$

IF P IS TOO SMALL,

PREVIOUS MODEL STILL SAYS $P \to C$

WHEREAS WE MIGHT EXPECT $P \to 0$

IE. AS P GETS SMALLER k GOES NEGATIVE $\frac{dP}{dt} =$

$$\frac{dP}{dt} = (mP - b)P \qquad m, b > 0$$

$$= m\left(P - \frac{b}{m}\right)P \qquad P_s = \frac{b}{m} > 0$$

$$= m(P - P_s)P$$

SIMILAR TO LAST DE

$(m, C) \to (-m, P_s)$

$$P = \frac{P_s P_0}{P_0 + (P_s - P_0)e^{mP_s t}} \quad (m, P_s > 0)$$

AS $t \to \infty$, $P \to \dfrac{P_s P_0}{P_0 \pm \infty} \to 0$

$k(P) = -mP + b$ IF $b > 0$

$k(P) > 0$ FOR

$k(P) = -mP - b < 0$

FOR ALL P

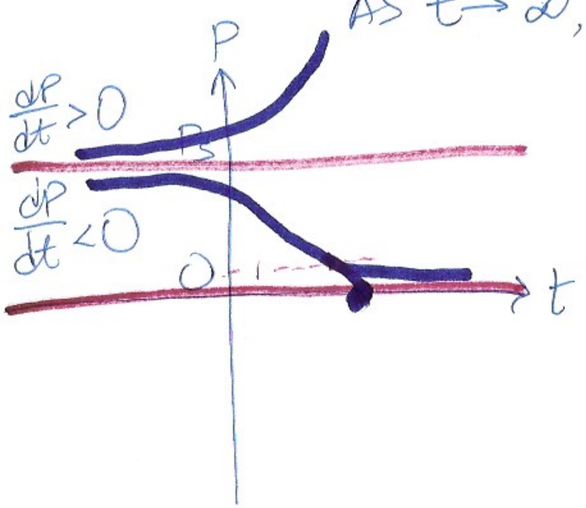
$k(P) = mP - b$

$m, b > 0$

$k(P) = 0$ @ $P = \frac{b}{m}$

$k(P) < 0$ IF $P < \frac{b}{m}$

$k(P) > 0$ IF $P > \frac{b}{m}$



$$\frac{dP}{dt} = m(P - P_s)P = 0 \text{ @ } P = 0, P_s$$

IF $P_0 > P_s$, PHASE PORTRAIT SAYS $P \to \infty$ AS $t \to \infty$

$P_s - P_0 < 0$

$$P = \frac{P_s P_0}{P_0 + (P_s - P_0)e^{mP_s t}} \qquad m, P_s > 0$$

$$P_0 + (P_s - P_0)e^{mP_s t} = 0$$

$$P_0 = (P_0 - P_s)e^{mP_s t} \qquad P_0 - P_s > 0$$

$$\frac{P_0}{P_0 - P_s} = e^{mP_s t} \qquad \frac{P_0}{P_0 - P_s} > 0$$

$$t = \frac{1}{mP_s} \underbrace{\ln \frac{P_0}{P_0 - P_s}}_{>0} \qquad P_0 > P_0 - P_s > 0$$

$$\frac{P_0}{P_0 - P_s} > 1$$

AT SOME $t > 0$

$$P_0 + (P_s - P_0)e^{mP_s t} = 0$$

IE, $P(t)$ HAS A VERTICAL ASYMPTOTE

Population goes to ∞ IN FINITE TIME